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Study of the evaporation of a droplet in its stagnant vapor by asymptotic matching

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Abstract—We study the evaporation of a droplet in its own stagnant vapor. Using three successive asymptotic expansions we can deduce an analytic expression for the drop radius. The nondimensionalized numbers involved in the expansion are the Mach number, the ratio of the thermal diffusivities and the Stefan number. The results obtained improve the previous ones derived with the so called quasi-permanent hypothesis. The comparison with numerical computation shows that the expression obtained by singular perturbation is a good approximation. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

The vaporization of a droplet has been considered as an important problem for spray combustion and the cooling of a hot gas stream. The first studies based on continuum hypothesis were summed up in the work of Fuks [1] in 1959. We will consider here, the vaporization of a drop in its own vapor. It has been known for a long time that for a spherical drop of liquid evaporating in its vapor or burning, the square of the radius is a linear function of time. This law has been verified experimentally see Williams [2] or Gyarmarthy [3], and has been deduced theoretically [4, 5, 6], even in recent studies see Kruz [7]. The theoretical deduction has been always based on the 'quasi permanent' assumption. The authors supposed that it is legitimate to consider that the temperature inside the drop depends upon the time only as a function of the droplet radius. This hypothesis is valid only for a long time period. In this paper we will consider again this old problem, but on the basis of asymptotic matching, we will show that it is possible to obtain a solution valid for all time, as the solution of a boundary layer problem, and to evaluate the correctness of the quasi-permanent hypothesis. Three expansions will be necessary. The first one involves the Mach number M, this number is the smallest, the second one involves the number ε ratio of the thermal diffusivities of the two media, and the third involves the Stefan number S. We will develop the two last expansions in ε and S up to first-order in ε and to second- and third-order in S. These two expansions will be singular. Using the matching rules of Van Dykes [8], we will obtain a corrected R^2 law better than the one obtained with the quasi-permanent assumption. The evaporation time, i.e. when the drop radius R vanishes, is obtained with more accuracy, and can be compared with the one computed numerically.

Section 2 is devoted to the notations and the position of the problem, the interface is considered as an infinitely thin surface and the temperature is continuous across the interface. In this paragraph we introduce the different nondimensional numbers used in the asymptotic analysis. Section 3 is devoted to the asymptotic analysis. We derive an expansion in the Stefan number for the square radius of the drop. The expansion is singular, therefore we need an inner and outer approximation that can be matched to obtain an explicit expression. This expression generalizes the one known previously, and we show that the relative correction is proportional to the Stefan number. In Section 4 we numerically solve a part of the problem and test the accuracy of the approximation, and we show that the second-order approximation gives a good expression for the drop radius.

2. POSITION OF THE PROBLEM

2.1. *Equations of the problem*

For the sake of simplicity, we consider a pure liquid in its own vapor, both of the fluids are supposed to be Newtonian fluids. As usual when a liquid is in contact with its vapor, the interfacial tension will be neglected. The gravitation is not considered here and we suppose that at the initial time the drop is spherical, and that all the quantities: densities, velocities and temperatures have spherical symmetry, in such a way that the problem remains spherical.

The vapor is in the exterior of the sphere occupied by the liquid. In the following the subscript k means v for vapor and l for liquid: ρ_k density, $V_k(r,t)$ radial velocity, $H_k(r,t)$ enthalpy, $P_k(r,t)$ pressure, $T_k(r,t)$ temperature, C_{vk} heat capacity, k_k heat conductivity, μ_k viscosity, $R(t)$ radius of the drop, L latent heat, T_{ev} evaporation temperature. The equations to be solved are the following (see Delhaye [9]):

Conservation of mass inside the fluids and at the interface

NOMENCLATURE

C_v	heat capacity
H	enthalpy
L	latent heat
M	Mach number
$\mathscr{P}$	square of radius
P	pressure
R	radius of the drop
Re	Reynolds number
S	Stefan number
T_k	temperature
T_{ev}	evaporation temperature
V	radial velocity
a	thermal diffusivity
k	heat conductivity
r	local radius of sphere
t	time.

Greek symbols

θ	temperature
ε	ratio of thermal diffusivities
ρ	density
μ	viscosity
σ	radius (Landau local radius)
τ	time (Landau time).

Subscripts

l	liquid
v	vapor
k	l or v
ev	evaporation
d	initial
0	reference.

$$\frac{\partial\rho_k}{\partial t}+\frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho_k V_k)=0 \tag{1}$$

$$\rho_l(V_l-\dot{R})=\rho_v(V_v-\dot{R})\quad \text{at } r=R(t). \tag{2}$$

where $\dot{R}$ denotes the time derivative of R.

The balance of momentum inside the fluids and at the interface

$$\rho_k\frac{dV_k}{dt}=-\frac{\partial P_k}{\partial r}+\frac{4}{3}\mu_k\left(\frac{\partial^2 V_k}{\partial r^2}+\frac{2}{r}\frac{\partial V_k}{\partial r}-\frac{2}{r^2}V_k\right) \tag{3}$$

$$\begin{aligned}P_v-P_l={}&-\rho_v(V_v-\dot{R})V_v+\rho_l(V_l-\dot{R})V_l\\&+\frac{4}{3}\mu_v\left(\frac{\partial V_v}{\partial r}-\frac{V_v}{r}\right)-\frac{4}{3}\mu_l\left(\frac{\partial V_l}{\partial r}-\frac{V_l}{r}\right)\text{at } r=R(t).\end{aligned} \tag{4}$$

The balance of energy inside the fluids and at the interface

$$\begin{aligned}\rho_k C_{vk}\frac{dT_k}{dt}={}&k_k\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_k}{\partial r}\right)\\&-P_k\frac{1}{r^2}\frac{\partial}{\partial r}(r^2V_k)+\frac{4}{3}\mu_v\left(\frac{\partial V_k}{\partial r}-\frac{V_k}{r}\right)^2\end{aligned} \tag{5}$$

$$\left.\begin{aligned}k_l\frac{\partial T_l}{\partial r}-k_v\frac{\partial T_v}{\partial r}={}&\rho_l(V_l-\dot{R})(H_l-H_v)\\&+\dot{R}(P_l-P_v)+\frac{1}{2}\rho_l(V_l-\dot{R})(V_l^2-V_v^2)\\&-\frac{4}{3}\mu_l\left(\frac{\partial V_l}{\partial r}-\frac{V_l}{r}\right)V_l\\&+\frac{4}{3}\mu_v\left(\frac{\partial V_v}{\partial r}-\frac{V_v}{r}\right)V_v\end{aligned}\right\}\text{at } r=R(t). \tag{6}$$

At the interface we suppose that the temperature is continuous

$$T_l=T_v=T_{ev}\quad\text{at } r=R(t). \tag{7}$$

We need the conditions at infinity and the initial condition

$$\left.\begin{aligned}P_v(\infty,t)&=P_v^{\infty}=\text{cste}\\V_v(\infty,t)&=0\\T_v(\infty,t)&=T_{\infty}=\text{cste}\end{aligned}\right\} \tag{8}$$

$$\left.\begin{aligned}R(0)&=R_d\\V_l(r,0)&=V_l^d(r)\\V_v(r,0)&=V_v^d(r)\\P_v(r,0)&=P_v(r)\end{aligned}\right\}\quad\text{and}\quad\left.\begin{aligned}T_l(r,0)&=T_l^d(r)\\T_v(r,0)&=T_v^d(r)\end{aligned}\right\}. \tag{9}$$

We suppose that all the thermodynamical parameters, i.e. the evaporation temperature T_{ev}, the latent heat L, the heat conductivities and capacities C_{vk} and k_k, and the viscosities μ_k are independent of pressure and temperature. The fact that the latent heat is considered as a constant is consistent with the assumption that the interface is infinitely thin. The liquids are generally weakly compressible, then we can consider that ρ_l is constant, therefore the conservation of mass (1) gives

$$V_l=0, \tag{10}$$

which simplifies relations (1)–(6). The conservation of momentum (3) in the fluids now reads $\partial P_l/\partial r=0$, and therefore the pressure in the fluid is constant, i.e.

$$P_l=\text{cst}(t). \tag{11}$$

The vapor is considered as a perfect gas

$$P_{\mathrm{v}} = R_{\mathrm{g}}\rho_{\mathrm{vl}}T_{\mathrm{v}}. \tag{12}$$

2.2. *Reduction*

Let us nondimensionalize the simplified relations [taking relations (10), (11) into account]. Let R_{d}, the initial radius of the drop, be the length reference, T_{ev}, the evaporation temperature, is the temperature reference, and if the pressure reference is P_{v}^{∞}, we choose ρ_0 as the reference density of the vapor given by the relation $P_{\mathrm{v}}^{\infty} = R_{\mathrm{g}}\rho_0 T_{\mathrm{ev}}$. If we denote by $a_{\mathrm{k}} = k_{\mathrm{k}}/\rho_{\mathrm{k}}C_{\mathrm{vk}}$ the thermal diffusivities, we can associate to a_{l} the reference time t_0 defined by $t_0 = R_{\mathrm{d}}^2/a_{\mathrm{l}}$, and the reference velocity V_0 defined by $V_0 = R_{\mathrm{d}}/t_0$. With these definitions in mind, we can proceed to the following change of variables:

$$r \to \frac{r}{R_{\mathrm{d}}} \quad R \to \frac{R}{R_{\mathrm{d}}} \quad t \to \frac{t}{t_0} \quad \rho_{\mathrm{v}} \to \frac{\rho_{\mathrm{v}}}{\rho_0} \quad \rho_{\mathrm{l}} \to \frac{\rho_{\mathrm{l}}}{\rho_0}$$

$$V_{\mathrm{v}} \to \frac{V_{\mathrm{v}}}{V_0} \quad P_{\mathrm{v}} \to \frac{P_{\mathrm{v}}}{P_{\mathrm{v}}^{\infty}}$$

$$T_{\mathrm{l}} \to \frac{T_{\mathrm{l}} - T_{\mathrm{ev}}}{T_{\mathrm{l}}^{\mathrm{d}}(0) - T_{\mathrm{ev}}} \quad T_{\mathrm{v}} \to \frac{T_{\mathrm{v}} - T_{\mathrm{ev}}}{T_{\mathrm{v}}^{\mathrm{d}}(\infty) - T_{\mathrm{ev}}}.$$

The reader will have noticed that in spite of the change of variables, we have kept the same symbols for the quantities involved in the different equations which become:

balance of mass

$$\frac{\partial \rho_{\mathrm{v}}}{\partial t} + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\rho_{\mathrm{v}}V_{\mathrm{v}}) = 0 \quad \text{for } r > R(t) \tag{13}$$

$$V_{\mathrm{v}}(R,t) = \left(1 - \frac{\rho_{\mathrm{l}}}{\rho_{\mathrm{v}}}\right)\dot{R}(t) \quad \text{for } r = R(t)\,; \tag{14}$$

balance of momentum

$$\rho_{\mathrm{v}}\frac{\mathrm{d}V_{\mathrm{v}}}{\mathrm{d}t} = -\frac{1}{\gamma M^2}\frac{\partial P_{\mathrm{v}}}{\partial r}$$

$$+\frac{1}{Re}\frac{4}{3}\left(\frac{\partial^2 V_{\mathrm{v}}}{\partial r^2} + \frac{2}{r}\frac{\partial V_{\mathrm{v}}}{\partial r} - \frac{2}{r^2}V_{\mathrm{v}}\right) \quad \text{for } r > R(t) \tag{15}$$

$$K(P_{\mathrm{v}} - P_{\mathrm{l}}) = \frac{\partial V_{\mathrm{v}}}{\partial r}(R,t) - \left(1 - \frac{\rho_{\mathrm{l}}}{\rho_{\mathrm{v}}}\right)$$

$$\times \dot{R}(t)\left(-\frac{3}{4}\rho_{\mathrm{l}}Re\,\dot{R} + \frac{1}{R}\right) \quad \text{for } r = R(t)\,; \tag{16}$$

balance of energy

$$\frac{\partial T_{\mathrm{l}}}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_{\mathrm{l}}}{\partial r}\right) \quad \text{for } r < R(t) \tag{17}$$

$$\varepsilon\rho_{\mathrm{v}}\frac{\mathrm{d}T_{\mathrm{v}}}{\mathrm{d}t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T_{\mathrm{v}}}{\partial r}\right) - \frac{\gamma-1}{J-1}\varepsilon P_{\mathrm{v}}\frac{1}{r^2}\frac{\partial}{\partial r}(r^2V_{\mathrm{v}})$$

$$+\frac{4}{3}\frac{\gamma(\gamma-1)}{J-1}\frac{\varepsilon}{Re}M^2\left(\frac{\partial V_{\mathrm{v}}}{\partial r} - \frac{V_{\mathrm{v}}}{r}\right)^2 \quad \text{for } r > R(t) \tag{18}$$

$$\frac{\partial T_{\mathrm{l}}}{\partial r} + F\frac{\partial T_{\mathrm{v}}}{\partial r} = -\frac{1}{S}\dot{R} + G\left(1 - \left(\frac{\rho_{\mathrm{l}}}{\rho_{\mathrm{v}}}\right)^2\right)\dot{R}^3$$

$$+\frac{\rho_{\mathrm{l}}}{\rho_{\mathrm{v}}}H\left[\frac{\partial V_{\mathrm{v}}}{\partial r} - \left(1 - \frac{\rho_{\mathrm{l}}}{\rho_{\mathrm{v}}}\right)\frac{\dot{R}}{R}\right]\dot{R} \quad \text{for } r = R(t). \tag{19}$$

At the interface the temperature continuity reads

$$T_{\mathrm{l}} = T_{\mathrm{v}} = 0. \tag{20}$$

The state equation for the vapor becomes

$$P_{\mathrm{v}} = \rho_{\mathrm{v}}[T_{\mathrm{v}}(J-1)+1]. \tag{21}$$

The following dimensionless numbers appear:

the Mach number

$$M = \frac{V_0}{\sqrt{\gamma R_{\mathrm{g}}T_{\mathrm{ev}}}}, \quad \text{where } \gamma = \frac{C_{\mathrm{vp}}}{C_{\mathrm{vv}}}$$

the Reynolds number

$$Re = \frac{\rho_0 V_0 R_{\mathrm{d}}}{\mu_{\mathrm{v}}}$$

the reduced density:

$$\rho_{\mathrm{l}} = \frac{\rho_{\mathrm{l}}}{\rho_0}$$

the ratio of the thermal diffusivities $\varepsilon = \dfrac{a_{\mathrm{l}}}{a_{\mathrm{v}}}$

the Stefan number $\quad S = \dfrac{C_{\mathrm{vl}}(T_{\mathrm{ev}} - T_{\mathrm{l}}^{\mathrm{d}})}{L}.$

Some other numbers appear defined by

$$F = \frac{k_{\mathrm{v}}}{k_{\mathrm{l}}}\varphi, \quad \text{with } \varphi = \frac{T_{\mathrm{v}}^{\mathrm{d}} - T_{\mathrm{ev}}}{T_{\mathrm{ev}} - T_{\mathrm{l}}^{\mathrm{d}}} \quad G = \frac{\gamma(\gamma-1)}{2(J-1)}F\rho_{\mathrm{l}}\varepsilon M^2$$

$$H = \frac{G}{\rho_{\mathrm{l}}Re} \quad J = \frac{T_{\mathrm{v}}^{\mathrm{d}}}{T_{\mathrm{ev}}} \quad K = \frac{3}{4}\frac{Re}{\gamma M^2}.$$

The numbers ε and Re can be related using the Prandtl number Pr by $\varepsilon = Pr\,Re/\gamma$.

The initial conditions now become

$$\begin{cases} R(0) = 1 \\ V_{\mathrm{v}}(r,0) = V_{\mathrm{v}}^{\mathrm{d}}(r) \\ P_{\mathrm{v}}(r,0) = P_{\mathrm{v}}(r) \\ T_{\mathrm{l}}(r,0) = T_{\mathrm{l}}^{\mathrm{d}}(r) \\ T_{\mathrm{v}}(r,0) = T_{\mathrm{v}}^{\mathrm{d}}(r) \end{cases} \tag{22}$$

the boundary conditions are

$$\begin{cases} P_{\mathrm{v}}(\infty,t) = 1 \\ V_{\mathrm{v}}(\infty,t) = 0. \\ T_{\mathrm{v}}(\infty,t) = 1 \end{cases}$$

Let us evaluate the different values of these nondimensional numbers. We consider a drop of water with $R_{\mathrm{d}} = 1$ mm, $T_{\infty} = 300°\mathrm{C}$, $T_{\mathrm{l}}^{\mathrm{d}}(0) = 60°C$, $P_{\infty} = 1$ atm. The physical tables give: $k_{\mathrm{l}} = 0.68$ W m^{-1} K^{-1}, $k_{\mathrm{v}} = 0.044$ W m^{-1} K^{-1}, $\rho_{\mathrm{l}} = 1000$ kg m^{-3}, $\rho_{\mathrm{v}} = 0.37$

kg m^{-3}, $\mu_l = 5\cdot 10^{-4}$ kg m^{-1} s^{-1}, $\mu_v = 2\cdot 10^{-5}$ kg m^{-1} s^{-1}, $C_{vl} = 4180$ J kg^{-1} K^{-1}, $C_{vv} = 2000$ J kg^{-1} K^{-1}. $L = 2250$ J kg^{-1} K^{-1}. With these values the different numbers are: $M = 3.5\times 10^{-7}$, $1/\rho_l = 3.7\times 10^{-4}$, $Re = 3.02\times 10^{-3}$, $\varepsilon = 2.7\times 10^{-3}$, $S = 7\times 10^{-2}$, $\gamma = 1.29$, $Pr = 1.1$, $F = 0.32$, $J = 1.54$, $K = 1.4\times 10^{10}$, $H \approx G = 9.96\times 10^{-14}$.

The parameters of the first line are small. They are placed in equations (15), (16) and (18), in such positions that boundary layers can appear. This fact, which we will illustrate later, must be considered in any numerical study of this system of equations. To handle the analytical solution, we will proceed to asymptotic expansions. But as there are several small parameters the expansions will be done in the order of increasing magnitude of the parameters. The first expansion will be made with respect to the Mach number.

3. ASYMPTOTIC ANALYSIS

As we indicated before, we are going to expand the solution with respect to different parameters. However the domains of the partial differential equations, defined by the radius $R(t)$ depend also on the parameters, because $R(t)$ is one of the unknowns of the problem. Therefore, we must carry out Landau transformation, which makes the domain a fixed one. That is, we consider the change of variables defined by

$$\left|\begin{matrix} t \\ r \end{matrix}\right. \to \left|\begin{matrix} \tau = t \\ \sigma = \dfrac{r}{R(t)} \end{matrix}\right. . \qquad (23)$$

In these new variables, the drop is parametrized by $\sigma \in [0, 1[$. We set $\theta(\sigma, \tau) = T(\sigma R(t), t)$, the temperature in the different domains, and we keep the same symbols for the velocity and density. Now equations (13)–(21) become:

(1) Mechanical equations
outside the drop, i.e. for $\sigma \geqslant 1$

$$\frac{\partial \rho_v}{\partial \tau} - \frac{\dot{R}}{R}\sigma\frac{\partial \rho_v}{\partial \sigma} + \frac{1}{R}\left(\frac{1}{\sigma^2}\frac{\partial}{\partial \sigma}(\sigma^2 \rho_v V_v)\right) = 0 \qquad (24)$$

$$\frac{\partial(\rho_v V_v)}{\partial \tau} - \frac{\dot{R}}{R}\sigma\frac{\partial(\rho_v V_v)}{\rho\sigma} + \frac{1}{R}\left(\frac{1}{\sigma^2}\frac{\partial}{\partial \sigma}(\sigma^2 \rho_v V_v^2)\right)$$
$$+ \frac{1}{\gamma M^2 R}\frac{\partial P_v}{\partial r} - \frac{1}{Re}\frac{4}{3R^2}\left(\frac{\partial^2 V_v}{\partial \sigma^2} + \frac{2}{\sigma}\frac{\partial V_v}{\partial \sigma} - \frac{2}{\sigma^2}V_v\right) = 0 \qquad (25)$$

$$P_v = \rho_v[\theta_v(J-1)+1] \qquad (26)$$

$$\left.\begin{matrix} P_v(\infty, \tau) = 1 \\ V_v(\infty, \tau) = 0 \end{matrix}\right\}, \qquad (27)$$

at the surface of the drop, i.e. for $\sigma = 1$

$$V_v(1, \tau) = \left(1 - \frac{\rho_l}{\rho_v}\right)\dot{R}(t) \qquad (28a)$$

$$K(P_v - P_l) = \frac{\partial V_v}{\partial \sigma}(1, \sigma)$$
$$- \left(1 - \frac{\rho_l}{\rho_v}\right)\dot{R}(t)\left(-\frac{3}{4}\rho_l Re\, R\dot{R} + 1\right); \qquad (28b)$$

(2) Thermal equations
inside the drop, i.e. for $\sigma \leqslant 1$

$$\frac{\partial \theta_l}{\partial \tau} - \frac{\dot{R}}{R}\sigma\frac{\partial(\theta_l)}{\partial \sigma} = \frac{1}{R^2\sigma^2}\frac{\partial}{\partial \sigma}\left(\sigma^2\frac{\partial \theta_l}{\partial \sigma}\right), \qquad (29)$$

outside the drop, i.e. for $\sigma \geqslant 1$

$$\varepsilon\rho_v\left(\frac{\partial \theta_v}{\partial \tau} - \frac{\dot{R}}{R}\sigma\frac{\partial \theta_v}{\partial \sigma} + \frac{V_v}{R}\frac{\partial \theta_v}{\partial \sigma}\right)$$
$$= \frac{1}{R^2\sigma^2}\frac{\partial}{\partial \sigma}\left(\sigma^2\frac{\partial \theta_v}{\partial \sigma}\right)$$
$$- \frac{\gamma - 1}{J-1}\varepsilon\frac{P_v}{R}\frac{1}{\sigma^2}\frac{\partial}{\partial \sigma}(\sigma^2 V_v)$$
$$+ \frac{\gamma(\gamma-1)}{J-1}\frac{\varepsilon}{Re}M^2\frac{4}{3R^2}\left(\frac{\partial V_v}{\partial \sigma} - \frac{V_v}{\sigma}\right)^2 \qquad (30)$$

$$\theta_v(\infty, t) = 1, \qquad (31)$$

at the surface of the drop, i.e. for $\sigma = 1$

$$\frac{\partial \theta_l}{\partial \sigma} + F\frac{\partial \theta_v}{\partial \sigma} = -\frac{1}{S}\dot{R}R + G\left(1 - \left(\frac{\rho_l}{\rho_v}\right)^2\right)R\dot{R}^3$$
$$+ \frac{\rho_l}{\rho_v}H\left[\frac{\partial V_v}{\partial \sigma} - \left(1 - \frac{\rho_l}{\rho_v}\right)\frac{\dot{R}}{R}\right]R\dot{R} \qquad (32)$$

$$\theta_l = \theta_v = 0, \qquad (33)$$

initial conditions

$$R(0) = 1$$
$$V_v(\sigma, 0) = V_v^{\mathrm{d}}(\sigma)$$
$$p_v(\sigma, 0) = P_v(\sigma)$$
$$\theta_l(\sigma, 0) = \theta_l^{\mathrm{d}}(\sigma)$$
$$\theta_v(\sigma, 0) = \theta_v^{\mathrm{d}}(\sigma). \qquad (34)$$

3.1. *Mach number expansion*

For the sake of completeness, we will begin with an expansion in Mach number. It is well known that in a reactive weakly compressible medium, the perturbation is singular at the initial time, see Dwyer [10] for example. We obtain a similar result—for the outer expansion (long time expansion) the dynamic process (balance of momentum) is completely driven by the thermal field. We will not calculate the inner expansion (short time expansion) and match it to the outer expansion, because the value of the Mach number is not of the same magnitude as the values of the other parameters and where the asymptotic limit is reached in a very short time on the one hand, and on the other hand there is no evidence in the experimental literature of such an effect, see Montluçon [4]. We calculate the first order only and set

$$V_{\rm v}(\sigma,\tau,M) = V^0(\sigma,\tau)+O(M)$$

$$P_{\rm v}(\sigma,\tau,M) = P_{\rm v}^0(\sigma,\tau)+P_{\rm v}^2(\sigma,\tau)M^2+O(M^2)$$

$$\theta_{\rm v}(\sigma,\tau,M) = \theta_{\rm v}^0(\sigma,\tau)+O(M)$$

$$P_{\rm l}(\sigma,\tau,M) = P_{\rm l}^0(\sigma,\tau)+P_{\rm l}^2(\sigma,\tau)M^2+O(M^2)$$

$$\theta_{\rm l}(\sigma,\tau,M) = \theta_{\rm l}^0(\sigma,\tau)+O(M)$$

$$\rho_{\rm v}(\sigma,\tau,M) = \rho_{\rm v}^0(\sigma,\tau)+O(M)$$

$$R(\tau,M) = R^0(\tau)+O(M).$$

If we introduce these expressions into relations (24)–(27), and collect the first-order terms, we obtain the following mechanical equations

$$\frac{\partial\rho_{\rm v}^0}{\partial\tau}-\frac{\dot R^0}{R^0}\sigma\frac{\partial\rho_{\rm v}^0}{\partial\sigma}+\frac{1}{R^0}\frac{1}{\sigma^2}\frac{\partial}{\partial\sigma}(\sigma^2\rho_{\rm v}^0V^0)=0. \quad (35)$$

The balance of momentum reduces to

$$\frac{\partial P_{\rm v}^0}{\partial\sigma}=0,$$

therefore the pressure depends only on τ and is uniform inside the vapor

$$P_{\rm v}^0=P_{\rm v}^0(\tau) \quad (36)$$

$$\rho_{\rm v}^0=\frac{P_{\rm v}^0(\tau)}{\theta_{\rm v}^0(J-1)+1} \quad (37)$$

$$V^0(1,\tau)=\left(1-\frac{\rho_{\rm l}}{P_{\rm v}^0(\tau)}\right)\dot R^0. \quad (38)$$

The boundary conditions read

$$\left.\begin{aligned}P_{\rm v}^0(\infty,\tau)&=1\\ V^0(\infty,\tau)&=0\end{aligned}\right\}. \quad (39)$$

So the pressure is uniform in the vapor at first order in M. If the initial pressure is not uniform, a transient regime must take place at the initial time with the propagation of pressure waves. For the sake of simplicity and for the reasons invoked before we will not study this transient phase and we will suppose an initial uniform pressure. Therefore, because of the boundary condition, the reduced pressure is equal to one. And the pressure inside the drop is obtained from the jump condition (28b) and (36)

$$P_{\rm l}^0(\sigma,\tau)=1.$$

Taking equation (36) into account in the jump condition (28b) we obtain at second order in M

$$P_{\rm v}^2(1,\tau)-P_{\rm l}^2(1,\tau)=\left[\frac{\partial V^0}{\partial\sigma}\right.$$
$$\left.-\left(1-\frac{3}{4}\rho_{\rm l}Re\,R^0\dot R^0\right)\left(1-\frac{\rho_{\rm l}}{P_{\rm v}^0(\tau)}\right)\dot R^0\right]\frac{4\gamma}{3Re\,R^0}. \quad (40)$$

The thermal relations give:
inside the drop, i.e. $\sigma<1$

$$\frac{\partial\theta_{\rm l}^0}{\partial\tau}-\frac{\dot R^0}{R^0}\sigma\frac{\partial\theta_{\rm l}^0}{\partial\sigma}=\frac{1}{R^{02}}\frac{1}{\sigma^2}\frac{\partial}{\partial\sigma}\left(\sigma^2\frac{\partial\theta_{\rm l}^0}{\partial\sigma}\right), \quad (41)$$

outside the drop, i.e. $\sigma>1$

$$\varepsilon\left[\frac{\partial\theta_{\rm v}^0}{\partial\tau}-\sigma\frac{\dot R^0}{R^0}\frac{\partial\theta_{\rm v}^0}{\partial\sigma}\right]=\frac{1}{R^{02}}\frac{1}{\sigma^2}\frac{\partial}{\partial\sigma}$$
$$\left(\sigma^2\frac{\partial\theta_{\rm v}^0}{\partial\sigma}\right)(\theta_{\rm v}^0(J-1)+1)-\varepsilon\frac{V^0}{R^0}\frac{\partial\theta_{\rm v}^0}{\partial\sigma}$$
$$-\varepsilon\frac{\gamma-1}{J-1}(\theta_{\rm v}^0(J-1)+1)\frac{1}{R^0}\frac{1}{\sigma^2}\frac{\partial}{\partial\sigma}(\sigma^2V^0), \quad (42)$$

at the interface, i.e. $\sigma=1$

$$\theta_{\rm l}^0(1,\tau)=\theta_{\rm v}^0(1,\tau)=0 \quad (43)$$

$$\frac{\partial\theta_{\rm l}^0}{\partial\sigma}+F\frac{\partial\theta_{\rm v}^0}{\partial\sigma}=-\frac{1}{S}R^0\dot R^0. \quad (44)$$

The boundary condition is

$$\theta_{\rm v}^0(\infty,\tau)=1 \quad (45)$$

and the initial conditions remain the same.

At this order, the viscosity plays no role in the dissipation; there is also a simple relation between velocity V^0 and thermal field $\theta_{\rm v}^0$. Let us eliminate $\rho_{\rm v}^0$ between relation (35) and relation (37), which yields

$$\frac{\theta_{\rm v}^0(J-1)+1}{R^0}\frac{1}{\sigma^2}\frac{\partial}{\partial\sigma}(\sigma^2V^0)$$
$$=(J-1)\left(\frac{-\sigma\dot R^0+V^0}{R^0}\frac{\partial\theta_{\rm v}^0}{\partial\sigma}+\frac{\partial\theta_{\rm v}^0}{\partial\tau}\right).$$

This relation can be combined with relation (42) to obtain

$$\frac{\partial}{\partial\sigma}(\gamma\varepsilon\sigma^2V^0)=\frac{J-1}{R^0}\frac{\partial}{\partial\sigma}\left(\sigma^2\frac{\partial\theta_{\rm v}^0}{\partial\sigma}\right).$$

This last relation can be integrated easily, and if we take into account the jump condition (38) we obtain

$$\varepsilon V^0=\frac{J-1}{R^0\gamma}\left(\frac{\partial\theta_{\rm v}^0}{\partial\sigma}-\frac{1}{\sigma^2}\left(\frac{\partial\theta_{\rm v}^0}{\partial\sigma}(1,\tau)-\frac{\gamma R^0\dot R^0}{J-1}(\varepsilon-\varepsilon\rho_{\rm l})\right)\right). \quad (46)$$

As a result of the velocity boundary condition (39), the temperature must satisfy the relation $\lim_{\sigma\to\infty}\partial\theta_{\rm v}^0/\partial\sigma=0$, which we then have to verify once the thermal field is known. The relation (46) is not usually satisfied by the initial conditions, which reinforces the fact that it will appear as a boundary layer at the initial time. The perturbation is singular, and therefore we ignore it for the sake of simplicity. Usually the mechanical problem is not considered, and the pressure is set to a constant in the vapor. At this stage, it remains to solve a thermal problem of which we recall the system of equations:

inside the drop, $\sigma < 1$,

$$\frac{\partial \theta_1^0}{\partial \tau} - \frac{\dot{R}^0}{R^0}\sigma\frac{\partial \theta_1^0}{\partial \sigma} = \frac{1}{R^{0^2}}\frac{1}{\sigma^2}\frac{\partial}{\partial \sigma}\left(\sigma^2\frac{\partial \theta_1^0}{\partial \sigma}\right), \tag{47}$$

outside the drop, $\sigma > 1$

$$\varepsilon\left[\frac{\partial \theta_v^0}{\partial \tau} + \frac{-\sigma\dot{R}^0 + V^0}{R^0}\frac{\partial \theta_v^0}{\partial \sigma}\right] = \frac{\theta_v^0(J-1)+1}{\gamma R^{0^2}}\frac{1}{\sigma^2}\frac{\partial}{\partial \sigma}\left(\sigma^2\frac{\partial \theta_v^0}{\partial \sigma}\right), \tag{48}$$

at the interface, $\sigma = 1$

$$\theta_1^0(1,\tau) = \theta_v^0(1,\tau) = 0 \tag{49}$$

$$\frac{\partial \theta_1^0}{\partial \sigma} + F\frac{\partial \theta_v^0}{\partial \sigma} = -\frac{1}{S}R^0\dot{R}^0, \tag{50}$$

boundary conditions

$$\theta_v^0(\infty,\tau) = 1. \tag{51}$$

Relation (48) has been deduced eliminating V^0, using equation (46), in the last term of equation (42). The influence of the mechanical problem appears only in the term containing the velocity V^0 in the left hand side of equation (48), and in appropriate initial conditions coming from the matching of an outer expansion [solution of the system (35)–(45)], and an inner expansion which still remains to be computed.

3.2. *Expansion in ε*

In fact the problem involves the small parameters ε and $1/\rho_1$. ε is small because the value of the liquid density is much greater than the one of the vapor. Therefore, these two parameters are related and we set $\varepsilon = \beta(1/\rho_1)$, where $\beta = k_1C_{vv}/k_vC_{v1}$. We set also $\bar{V} = \varepsilon V^0/R^0$, and $\mathscr{P} = R^{0^2}$. As there is no ambiguity we discard the index 0 in the previous system of equations, which can be written:

inside the drop, i.e. $\sigma < 1$

$$\frac{\partial \theta_1}{\partial \tau} - \frac{\dot{\mathscr{P}}}{2\mathscr{P}}\sigma\frac{\partial \theta_1}{\partial \sigma} = \frac{1}{\mathscr{P}}\frac{1}{\sigma^2}\frac{\partial}{\partial \sigma}\left(\sigma^2\frac{\partial \theta_1}{\partial \sigma}\right); \tag{52}$$

outside the drop, $\sigma > 1$

$$\varepsilon\left[\frac{\partial \theta_v}{\partial \tau} - \frac{\dot{\mathscr{P}}}{2\mathscr{P}}\sigma\frac{\partial \theta_v}{\partial \sigma}\right] = \frac{\theta_v(J-1)+1}{\gamma\mathscr{P}}\frac{1}{\sigma^2}\frac{\partial}{\partial \sigma}\left(\sigma^2\frac{\partial \theta_v}{\partial \sigma}\right) - \bar{V}\frac{\partial \theta_v}{\partial \sigma}, \tag{53}$$

where

$$\bar{V} = \frac{J-1}{\mathscr{P}\gamma}\left(\frac{\partial \theta_v}{\partial \sigma} - \frac{1}{\sigma^2}\left(\frac{\partial \theta_v}{\partial \sigma}(1,\tau) - \frac{\gamma\dot{\mathscr{P}}}{2(J-1)}(\varepsilon - \varepsilon\rho_1)\right)\right); \tag{54}$$

at the interface, $\sigma = 1$

$$\theta_1(1,\tau) = \theta_v(1,\tau) = 0 \tag{55}$$

$$\frac{\partial \theta_1}{\partial \sigma} + F\frac{\partial \theta_v}{\partial \sigma} = -\frac{1}{2S}\dot{\mathscr{P}}; \tag{56}$$

boundary conditions

$$\theta_v(\infty,\tau) = 1. \tag{57}$$

It is now evident from equation (53) (the ε being a factor of the time derivative) that the ε expansion is singular and there is a boundary layer for $\tau = 0$, thus we will proceed to the outer and inner expansion (the terms inner and outer do not mean inside and outside the drop, but mean short time and long time behaviour).

3.2.1. *Outer expansion in ε.* In order to carry out the expansion we set

$$\theta_v(\sigma,\tau,\varepsilon) = \theta_v^0(\sigma,\tau) + O(\varepsilon)$$

$$\theta_1(\sigma,\tau,\varepsilon) = \theta_1^0(\sigma,\tau) + O(\varepsilon)$$

$$\mathscr{P}(\tau,\varepsilon) = \mathscr{P}^0(\tau) + O(\varepsilon).$$

We could insert these expressions into the system (52)–(57), but we would obtain an equation for the vapor temperature for which the integration is difficult. It is preferable to obtain a relation between velocity and temperature at order 1, from which we will derive an explicit expression for the vapor temperature field. Directly from relations (35) and (37) we have

$$\frac{1}{\sigma^2}\frac{\partial}{\partial \sigma}\left(\frac{\sigma^2}{\theta_v(J-1)+1}\bar{V}\right) = \varepsilon\frac{J-1}{(\theta_v(J-1)+1)^2}\left(\frac{\partial \theta_v}{\partial \tau} - \frac{\dot{\mathscr{P}}}{2\mathscr{P}}\sigma\frac{\partial \theta_v}{\partial \sigma}\right). \tag{58}$$

Let us note that relation (54) shows that $\bar{V}(\rho,\tau,\varepsilon) = \bar{V}^0(\rho,\tau) + O(\varepsilon)$, then we obtain

$$\frac{\partial}{\partial \sigma}\left(\frac{\sigma^2}{\theta_v(J-1)+1}\bar{V}^0\right) = 0.$$

This relations implies $\bar{V}^0 = f(t)/\sigma^2(\theta_v^0(J-1)+1)$, where $f(t)$ can be determined with the jump condition at first order in Mach number: $\bar{V}(1,\tau) = (\varepsilon - \beta)(\dot{\mathscr{P}}/2\mathscr{P})$, which can be written at order one in ε: $\bar{V}^0(1,\tau) = -\beta(\dot{\mathscr{P}}^0/2\mathscr{P}^0)$; and finally we obtain

$$\tilde{V}^0 = -\beta \frac{\dot{\mathscr{P}}^0}{2\mathscr{P}^0} \frac{\theta_v^0(J-1)+1}{\sigma^2}. \qquad (59)$$

We are now able to obtain the temperature field.

(1) Outside the drop, or in the vapor, i.e. for $\sigma > 1$

Taking into account the previous relation (59) for the velocity, relation (53) gives at first order

$$\frac{1}{\sigma^2}\frac{\partial}{\partial \sigma}\left(\sigma^2 \frac{\partial \theta_v^0}{\partial \sigma}\right) = -\frac{\gamma\beta\dot{\mathscr{P}}^0}{2}\frac{1}{\sigma^2}\frac{\partial \theta_v^0}{\partial \sigma},$$

which can be integrated in

$$\sigma^2 \frac{\partial \theta_v^0}{\partial \sigma} + \frac{\gamma\beta\dot{\mathscr{P}}^0}{2}\theta_v^0 = h(\tau). \qquad (60)$$

Let us show that $h(\tau)$ is defined by

$$h(\tau) = \frac{\partial \theta_v^0}{\partial \sigma}(1,\tau). \qquad (61)$$

To integrate equation (60) let us set $\theta_v^* = \theta_v^0 - [2h(\tau)/\gamma\beta\dot{\mathscr{P}}^0]$. We see that $\theta_v^*(1,\tau) = -[2h(\tau)/\gamma\beta\dot{\mathscr{P}}^0]$ and with this notation in mind, we notice that equation (60) can be written $\sigma^2(\partial\theta_v^*/\partial\sigma) + (\gamma\beta\dot{\mathscr{P}}^0/2)\theta_v^* = 0$, which can be integrated in

$$\theta_v^0 = h(\tau)\frac{\exp\left[-\frac{\gamma\beta\dot{\mathscr{P}}^0}{2}\left(1-\frac{1}{\sigma}\right)\right]-1}{\frac{\gamma\beta\dot{\mathscr{P}}^0}{2}}. \qquad (62)$$

Using relation (57) we can deduce the unknown function $h(\tau)$, thus

$$h(\tau) = \frac{-\gamma\beta\dot{\mathscr{P}}^0/2}{\exp[-\gamma\beta\dot{\mathscr{P}}^0/2]-1}.$$

The temperature field in the vapor is given by

$$\theta_v^0 = \frac{\exp\left[-\frac{\gamma\beta\dot{\mathscr{P}}^0}{2}\left(1-\frac{1}{\sigma}\right)\right]-1}{\exp\left[-\frac{\gamma\beta\dot{\mathscr{P}}^0}{2}\right]-1}. \qquad (63)$$

If we inject into relation (54) the value of θ_v^0 given by equation (61), we will recover equation (63). The problem in the vapor is fully solved once the evolution of $\mathscr{P}^0$, the radius square, is known. But once again relation (63) written at initial time gives an *a priori* form of the initial field of temperature. This field, given by equation (63), has no reason to be the real initial field. Therefore a boundary layer can occur, and the perturbation is singular.

(2) Inside the drop or in the liquid, i.e. for $\sigma < 1$

We have to compute the thermal field of the vapor, for a long time. We must expand relations (52) and (57) in power of ε, which gives

$$\frac{\partial \theta_l^0}{\partial \tau} - \frac{\dot{\mathscr{P}}^0}{2\mathscr{P}^0}\sigma\frac{\partial \theta_l^0}{\partial \sigma} = \frac{1}{\mathscr{P}^0}\frac{1}{\sigma^2}\frac{\partial}{\partial \sigma}\left(\sigma^2\frac{\partial \theta_l^0}{\partial \sigma}\right). \qquad (64)$$

At the interface, i.e. for $\sigma = 1$

Taking equation (63) into account, equation (56) gives the boundary condition

$$\frac{\partial \theta_l^0}{\partial \sigma} = -F\frac{-\gamma\beta\dot{\mathscr{P}}^0/2}{\exp[-\gamma\beta\dot{\mathscr{P}}^0/2]-1} - \frac{1}{2S}\dot{\mathscr{P}}^0 \qquad (65)$$

$$\theta_l^0(1,\tau) = 0. \qquad (66)$$

If we want to sum up the situation, we have a first-order outer expansion $(\theta_l^0(\sigma,\tau,S), \theta_v^0(\sigma,\tau,S), \mathscr{P}^0(\tau;S))$, i.e. defined for long time, as a solution of the free boundary problem (64)–(66) for θ_l^0, and by equation (63) for θ_v^0, but with no appropriate initial conditions either in the liquid or in the vapor.

3.2.2. *Inner expansion in* ε. We are now in a position to obtain an inner expansion, for that purpose let us set

$$\tau_0 = \frac{\tau}{\varepsilon}, \quad \tilde{\mathscr{P}}(\tau_0) = \frac{\mathscr{P}(\tau)-1}{\varepsilon} \quad \text{and} \quad \tilde{\theta}(\sigma,\tau_0) = \theta(\sigma,\tau). \qquad (67)$$

With these new variables the system (52)–(57) can be written:

inside the drop or in the liquid, i.e. for $\sigma < 1$

$$\frac{\partial \tilde{\theta}_l}{\partial \tau_0} - \varepsilon\frac{\dot{\tilde{\mathscr{P}}}}{2(1+\varepsilon\tilde{\mathscr{P}})}\sigma\frac{\partial\tilde{\theta}}{\partial\sigma} = \varepsilon\frac{1}{(1+\varepsilon\tilde{\mathscr{P}})}\frac{1}{\sigma^2}\frac{\partial}{\partial\sigma}\left(\sigma^2\frac{\partial\tilde{\theta}_l}{\partial\sigma}\right); \qquad (68)$$

outside the drop, or in the vapor, i.e. for $\sigma > 1$

$$\frac{\partial \tilde{\theta}_v}{\partial \tau_0} - \varepsilon\frac{\dot{\tilde{\mathscr{P}}}}{2(1-\varepsilon\tilde{\mathscr{P}})}\sigma\frac{\partial\tilde{\theta}_v}{\partial\sigma} = \frac{\tilde{\theta}_v(J-1)+1}{\gamma(1+\varepsilon\tilde{\mathscr{P}})}\frac{1}{\sigma^2}\frac{\partial}{\partial\sigma}\left(\sigma^2\frac{\partial\tilde{\theta}_v}{\partial\sigma}\right) - \tilde{V}\frac{\partial\tilde{\theta}_v}{\partial\sigma} \qquad (69)$$

with

$$\tilde{V} = \frac{J-1}{\gamma(1+\varepsilon\tilde{\mathscr{P}})}\left(\frac{\partial\tilde{\theta}_v}{\partial\sigma} - \frac{1}{\sigma^2}\left(\frac{\partial\tilde{\theta}_v}{\partial\sigma}(1,\tau_0) - \frac{\gamma}{J-1}(\varepsilon-\beta)\frac{\dot{\tilde{\mathscr{P}}}}{2}\right)\right); \qquad (70)$$

at the interface, i.e. for $\sigma = 1$

$$\tilde{\theta}_l(1,\tau_0) = \tilde{\theta}_v(1,\tau_0) = 0 \qquad (71)$$

$$\frac{\partial\tilde{\theta}_l}{\partial\sigma} + F\frac{\partial\tilde{\theta}_v}{\partial\sigma} = -\frac{1}{2S}\dot{\tilde{\mathscr{P}}}. \qquad (72)$$

The boundary condition is

$$\tilde{\theta}_v(\infty, \tau_0) = 1. \tag{73}$$

We must add the initial conditions

$$\tilde{\theta}_v(\sigma, 0) = \theta_v(\sigma, 0) \quad \text{and} \quad \tilde{\theta}_l(\sigma, 0) = \theta_l(\sigma, 0). \tag{74}$$

Now we look for an expansion of the following form:

$$\tilde{\theta}_v(\sigma, \tau_0, \varepsilon) = \tilde{\theta}_v^0(\sigma, \tau_0) + O(\varepsilon)$$

$$\tilde{\theta}_l(\sigma, \tau_0, \varepsilon) = \tilde{\theta}_l^0(\sigma, \tau_0) + O(\varepsilon)$$

$$\tilde{\mathscr{P}}(\tau_0, \varepsilon) = \tilde{\mathscr{P}}^0(\tau_0) + O(\varepsilon).$$

At first-order we obtain:
inside the drop or in the liquid, i.e. for $\sigma < 1$

$$\frac{\partial \tilde{\theta}_l^0}{\partial \tau_0} = 0; \tag{75}$$

outside the drop, or in the vapor, i.e. for $\sigma > 1$

$$\frac{\partial \tilde{\theta}_v^0}{\partial \tau_0} = \frac{\tilde{\theta}_v^0(J-1)+1}{\gamma} \frac{1}{\sigma^2} \frac{\partial}{\partial \sigma}\left(\sigma^2 \frac{\partial \tilde{\theta}_v^0}{\partial \sigma}\right) - \tilde{V}^0 \frac{\partial \tilde{\theta}_v^0}{\partial \sigma} \tag{76}$$

with

$$\tilde{V}^0 = \frac{J-1}{\gamma}\left(\frac{\partial \tilde{\theta}_v^0}{\partial \sigma} - \frac{1}{\sigma^2}\left(\frac{\partial \tilde{\theta}_v^0}{\partial \sigma}(1, \tau_0) + \frac{\gamma}{J-1}\beta \frac{\dot{\tilde{\mathscr{P}}}^0}{2}\right)\right); \tag{77}$$

at the interface, i.e. for $\sigma = 1$

$$\tilde{\theta}_l^0(1, \tau_0) = \tilde{\theta}_v^0(1, \tau_0) = 0 \tag{78}$$

$$\frac{\partial \tilde{\theta}_l^0}{\partial \sigma} + F \frac{\partial \tilde{\theta}_v^0}{\partial \sigma} = -\frac{1}{2S} \dot{\tilde{\mathscr{P}}}^0. \tag{79a}$$

The boundary condition is

$$\tilde{\theta}_v^0(\infty, \tau_0) = 1, \tag{80}$$

and the initial conditions are

$$\tilde{\theta}_v^0(\sigma, 0) = \theta_v(\sigma, 0) = \theta_v^i(\sigma) \tag{81}$$

$$\text{and} \quad \tilde{\theta}_l^0(\sigma, 0) = \theta_l(\sigma, 0) = \theta_l^i(\sigma).$$

Relation (75) implies that the temperature in the drop is constant, that is $\tilde{\theta}_l^0(\sigma, \tau_0) = \theta_l^i(\sigma)$ with the notation of the initial temperature fields given in equation (81). The Van Dyke [8] matching condition reads

$$\lim_{\tau \to 0} (\theta_l^0) = \lim_{\tau \to \infty} (\tilde{\theta}_l^0) \quad \text{in the fluid} \tag{82a}$$

$$\lim_{\tau \to 0} (\theta_v^0) = \lim_{\tau \to \infty} (\tilde{\theta}_v^0) \quad \text{in the vapor} \tag{82b}$$

$$\lim_{\tau \to 0} (\mathscr{P}^0) = \lim_{\tau_0 \to 0} (\tilde{\mathscr{P}}^0) = 1$$

$$\text{for the square of the radius.} \tag{82c}$$

At this order, the temperature in the fluid is steady and equal to the initial temperature. Therefore the temperature in the vapor and the radius are the only unknown quantities, indeed the temperature of the liquid can be substituted in equation (79a) to obtain a system of equations for the vapor depending only upon the unknowns, and we obtain

$$\dot{\tilde{\mathscr{P}}}^0 = -2S\left[\frac{\mathrm{d}\theta_l^i}{\mathrm{d}\sigma}(1) + F \frac{\partial \tilde{\theta}_v^0}{\partial \sigma}(1, \tau_0)\right]. \tag{79}$$

The system of equations for the inner problem is now complete.

3.3. *Derivation of the asymptotic expression of the radius*

Let us recall, at this stage the statement of the asymptotic problem we obtained:

(1) For a short time period the inner expansion is valid. The temperature inside the liquid is constant and equal to the initial temperature. The temperature in the vapor as well as the radius are solutions of the system of equations (76)–(81).

(2) For a long time period the outer expansion is valid. The temperature in the vapor is explicit as a function of the radius in relation (63). The temperature in the liquid and the radius are solutions of the system (64)–(66), with no initial conditions.

(3) The initial conditions for the outer expansion are obtained by the matching conditions (80).

At this order, the inner solution for the liquid is known, and the outer solution does not depend on the vapor temperature field, so the problem inside the liquid can theoretically be completely determined as well as the radius.

3.3.1. *Short time expansion.* The Stefan number S is small in order to obtain an analytical solution of the problem. It is tempting to proceed to an asymptotic expansion with respect to S, in the system (64)–(66). Let us set

$$\theta_l^0(\sigma, \tau, S) = \sum_{i=0}^{i} \theta_l^{0i}(\sigma, \tau) S^i$$

$$\mathscr{P}^0(\tau, S) = \sum_{i=0}^{i} \mathscr{P}^{0i}(\tau) S^i. \tag{83}$$

(i) At first order, equations (64)–(66) reduce to

$$\frac{\partial \theta_l^{00}}{\partial \tau} - \frac{\dot{\mathscr{P}}^{00}}{2\mathscr{P}^{00}} \sigma \frac{\partial \theta_l^{00}}{\partial \sigma} = \frac{1}{\mathscr{P}^{00}} \frac{1}{\sigma^2} \frac{\partial}{\partial \sigma}\left(\sigma^2 \frac{\partial \theta_l^{00}}{\partial \sigma}\right) \quad \text{for } \sigma < 1 \tag{84}$$

$$\theta_l^{00}(1, \tau) = 0 \tag{85}$$

$$\dot{\mathscr{P}}^{00}(\tau) = 0. \tag{86}$$

The initial conditions are the matching conditions (82a) and (82b). The solution of this system is

$$\mathscr{P}^{00}(\tau) = 1 \tag{87}$$

$$\theta_l^{00}(\sigma,\tau) = \sum_{n=1}^{\infty} A_n \frac{\sin(n\pi\sigma)}{\sigma}\exp(-(n\pi)^2\tau) \quad (88)$$

with

$$A_n = -2\int_0^1 \theta_l^i(u)\sin(n\pi u)u\,\mathrm{d}u. \quad (89)$$

(ii) At second order, we can deduce the evolution of the radius by

$$\frac{\partial\theta_l^{00}}{\partial\sigma}(1,\tau) = -F - \frac{1}{2}\dot{\mathscr{P}}^{01}(\tau).$$

Therefore, taking relation (88) and the matching initial condition (82c) into account, we obtain

$$\mathscr{P}^{01}(\tau) = -2\left[F\tau + \sum_{n=1}^{\infty} A_n \frac{(-1)^{n+1}}{n\pi}(\exp(-(n\pi)^2\tau)-1)\right] \quad (90)$$

which gives for the square of the radius

$$\mathscr{P}^0(\tau,S) = 1-2S\left[F\tau + \sum_{n=1}^{\infty} A_n \frac{(-1)^{n+1}}{n\pi} \times (\exp(-(n\pi)^2\tau)-1)\right] + O(S^2) \quad (91)$$

the initial velocity of the interface is given by

$$\dot{\mathscr{P}}^0(0) = -2\left[F - \sum_{n=1}^{\infty} n\pi(-1)^{n+1}A_n\right] = -2\left[F + \frac{\mathrm{d}\theta_l^i(1)}{\mathrm{d}\sigma}\right]. \quad (92)$$

The sign of this derivative indicates if a condensation or an evaporation occurs initially. In the case of an initial constant temperature inside the drop there is always a condensation at the beginning. These expansions are valid as long as the radius of the drop is different from zero and as long as the series defined in equation (83) converge. We can estimate from equation (91) a time of evaporation τ_{ev}^0, it is a solution of the equation $\mathscr{P}^0(\tau_{\mathrm{ev}}^0(S)) = 0$ given by

$$\tau_{\mathrm{ev}}^0(S) = \frac{1}{S}\frac{1}{2F} + \frac{1}{F}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n\pi}A_n + O(S). \quad (93)$$

The dominant term is $O(1/S)$, thus we need a solution valid for a long time period, the solution (91) being valid only for a short time period, as we will justify later. Let us remark that for all integer m

$$\theta_l^{00}(\sigma,\tau_{\mathrm{ev}}^0(S)) = o(S^m). \quad (94)$$

3.3.2. *Long time expansion.* In order to have a solution for a long time period we can dilate the time scale setting $\tau_1 = S\tau$; with $\tau_1 = O(1)$, thus $\tau = O(1/S)$. Let us consider $\hat{\theta}_l^0(\sigma,\tau_1) = \theta_l^0(\sigma,\tau)$ and $\hat{\mathscr{P}}^0(\tau_1) = \mathscr{P}^0(\tau)$. With this change of variables the equations (64)–(66) become

$$S\left(\frac{\partial\hat{\theta}_l^0}{\partial\tau_1} - \frac{\dot{\hat{\mathscr{P}}}^0}{2\hat{\mathscr{P}}^0}\sigma\frac{\partial\hat{\theta}_l^0}{\partial\sigma}\right) = \frac{1}{\hat{\mathscr{P}}^0}\frac{1}{\sigma^2}\frac{\partial}{\partial\sigma}\left(\sigma^2\frac{\partial\hat{\theta}_l^0}{\partial\sigma}\right) \quad \text{for } \sigma < 1 \quad (95)$$

$$\hat{\theta}_l^0(1,\tau_1) = 0 \quad (96)$$

$$\frac{\partial\hat{\theta}_l^0}{\partial\sigma}(1,\tau_1) = -F\frac{-\gamma S\beta\dot{\hat{\mathscr{P}}}^0/2}{\exp[-\gamma S\beta\dot{\hat{\mathscr{P}}}^0/2]-1} - \frac{1}{2}\dot{\hat{\mathscr{P}}}^0. \quad (97)$$

The dot means the derivation with respect to τ_1. As usual we can look for a solution of the following form:

$$\hat{\theta}_l^0(\sigma,\tau_1,S) = \sum_{i=0}^{\infty}\hat{\theta}_l^{0i}(\sigma,\tau_1)S^i$$

$$\hat{\mathscr{P}}^0(\tau_1,S) = \sum_{i=0}^{\infty}\hat{\mathscr{P}}^{0i}(\tau_1)S^i.$$

After some algebra one can show that at all order m

$$\hat{\theta}_l^{0m}(\sigma,\tau_1) = 0 \quad (98)$$

which is compatible with equation (94). Then equation (97) gives

$$-\frac{1}{2S}\dot{\hat{\mathscr{P}}}^0 - F\frac{-\gamma\beta\dot{\hat{\mathscr{P}}}^0/2}{\exp[-\gamma S\beta\dot{\hat{\mathscr{P}}}^0/2]-1} = o(S^m). \quad (99)$$

Relation (99) is an equation in $\dot{\hat{\mathscr{P}}}^0$ that we can solve

$$\dot{\hat{\mathscr{P}}}^0 = -2F\frac{\ln(1+\gamma\beta FS)}{\gamma\beta FS} + o(S^m). \quad (100)$$

At this scale of time the slope of the curve giving the radius square with respect to the time is constant and can be expressed in dimensionalized variables by

$$\dot{\hat{\mathscr{P}}}^0 = -2\frac{C_{\mathrm{vl}}k_{\mathrm{v}}}{C_{\mathrm{pv}}k_{\mathrm{l}}}\ln\left(1+\frac{C_{\mathrm{pv}}(T_{\mathrm{v}}^{\mathrm{d}}-T_{\mathrm{ev}})}{L}\right).$$

This relation has been derived by Williams [6] with the assumption of a quasi-stationary solution. Let us note that relation (100) provided a Nusselt number at the interface given by $1\ln(1+\gamma\beta FS)/S\gamma\beta FS$, which can be applied only for long time period. This relation has been given previously by Montluçon [4], also within the assumption of a quasi-stationary solution. Once integrated, equation (100) gives

$$\hat{\mathscr{P}}^0(\tau_1,S) = -2F\frac{\ln(1+\gamma\beta FS)}{\gamma\beta FS}\tau_1 + f(S), \quad (101)$$

where $f(S)$ is a function which remains to be determined. If we look for an expansion of $f(S)$ in the form

$$f(s) = \sum_{i=0} f^i S^i. \qquad (102)$$

The determination of this function can be realized using the matching rule of Van Dyke. To obtain the third order in equation (102), we match the expressions (101) and (90). At this order we have

$$f = 1 + 2S \sum_{n=1} \frac{(-1)^{n-1}}{n\pi} A_n + \gamma\beta F\left(\sum_{n=1} \frac{(-1)^{n+1}}{n\pi} A_n\right) S^2 + O(S^3). \qquad (103)$$

The evaporation time $\hat{\tau}_{ev} = S\tau_{ev}$ is modified, it is a solution of $\hat{P}^0(\hat{\tau}_{ev}(S), S) = 0$ and is given by

$$\hat{\tau}_{ev}(S) = \frac{2}{2F} \frac{\gamma\beta FS}{\ln(1+\gamma\beta FS)} f(S), \qquad (104)$$

the relative variation of the evaporation time calculated and matched at order two, i.e. f is given by equation (103) at order two, or without being matched, i.e. f is equal to one is

$$\frac{\Delta\tau_{ev}}{\tau_{ev}} = 2S \sum_{n=1} \frac{(-1)^{n+1}}{n\pi} A_n. \qquad (105)$$

In the case of a constant initial temperature $A_n = 2(-1)^{n+1}/n\pi$, therefore $\Delta\tau_{ev}/\tau_{ev} = (2/3)S$. It is possible to obtain an expansion of $f(S)$ at any order, but we will see in the next section that the second and third order give good approximations.

4. NUMERICAL COMPUTATION AND DISCUSSION

In order to know the precision of the asymptotic matching for the radius, we numerically solve the system of equations (64)–(66), but write in the moving domain, that is after performing the transformation inverse to the Landau transformation (23)

$$\left|\begin{aligned} & t = \tau \\ & r = \sigma R^0(t) \\ & T_1^0(r, t) = \theta_1^0(\sigma, \tau) \end{aligned}\right.$$

Let us quickly describe the numerical algorithm. The weak formulation can be written

$$\int_0^{R^0(t)} r^2 v \frac{\partial T_1^0}{\partial r} \mathrm{d}r + \int_0^{R^0(t)} r^2 \frac{\mathrm{d}v}{\mathrm{d}r} \frac{\partial T_1^0}{\partial r} \mathrm{d}r = R^0(t) v(R^0(t)) \left[-F \frac{-\gamma\beta\dot{\mathscr{P}}^0(t)}{\exp[-\gamma\beta\dot{\mathscr{P}}^0(t)] - 1} - \frac{1}{S}\dot{P}^0(t) \right]$$

for all functions $v(r)$. (106)

We choose a regular discretization of the interval $[0, R(0)]$, one-dimensional P1 elements. We note $v^j(r)$, the functions associated with this discretization. The temperature is discretized in

$$T_1^0 = \sum_{j=1}^{N} T^j(t) v^j.$$

The numerical system to be solved is

$$\left\{\begin{aligned} & \frac{\mathrm{d}T^j}{\mathrm{d}t} M_{ij}(R^0) + T^j R_{ij}(R^0) = N(\dot{\mathscr{P}}^0) v^j(R^0(t)) R^0(t) \\ & T^j = 0 \quad \text{for } r = R^0(t) \\ & + \text{initial conditions} \end{aligned}\right. \qquad (107)$$

with

$$\mathscr{P}^0(t) = (R^0(t))^2 \quad M_{ij}(R^0) = \int_0^{R^0(t)} r^2 v^i v^j \,\mathrm{d}r$$

$$R_{ij}(R^0) = \int_0^{R^0(t)} r^2 \frac{\mathrm{d}v^i}{\mathrm{d}r} \frac{\mathrm{d}v^j}{\mathrm{d}r} \mathrm{d}r$$

and

$$N(\dot{\mathscr{P}}^0) = -F \frac{-\gamma\beta\dot{\mathscr{P}}^0/2}{\exp[-\gamma\beta\dot{\mathscr{P}}^0/2] - 1} - \frac{1}{2S}\dot{\mathscr{P}}^0.$$

If Δt is the step time, we note $T^j(k\,\Delta t) = T^j_k$, and the linear system to solve, with an implicit scheme, can be written

$$\frac{T^j_{k+1} - T^j_k}{\Delta t} M_{ij}(R_{k+1}) + T^j_{k+1} R_{ij}(R_{k+1}) = R_{k+1} v^i(R_{k+1}) N\left(\frac{\mathscr{P}_{k+1} - \mathscr{P}_k}{\Delta t}\right). \qquad (108)$$

The value R_{k+1} is determined in order to obtain $T_{k+1} = 0$ for $r = R_{k+1}$.

Let us compare the results, the fluid and the vapor are initially at constant temperature, the evolution of the temperatures fields are given in Figs. 1 and 2.

In order to verify the asymptotic expansion in Mach number let us plot the velocity in the vapor (Fig. 3). We notice that the assumption used for the Mach number development is fulfilled.

Let us plot, on Fig. 4, the square of the radius computed numerically vs time, and the square of the

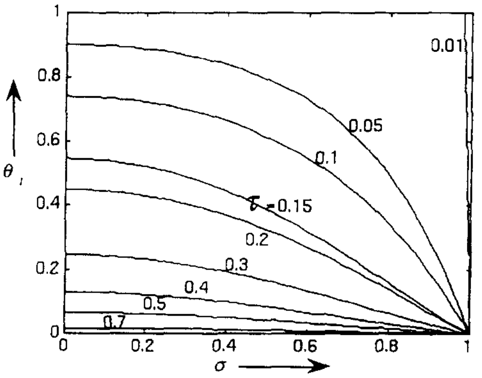

Fig. 1. Temperature in the drop, at different times τ.

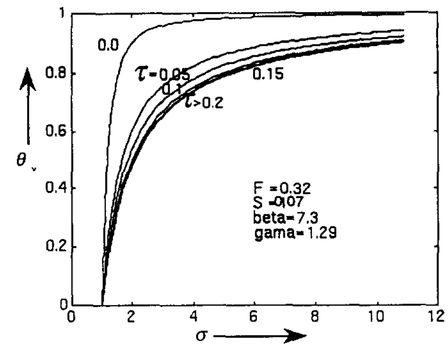

Fig. 2. Temperature in the vapor at different times τ.

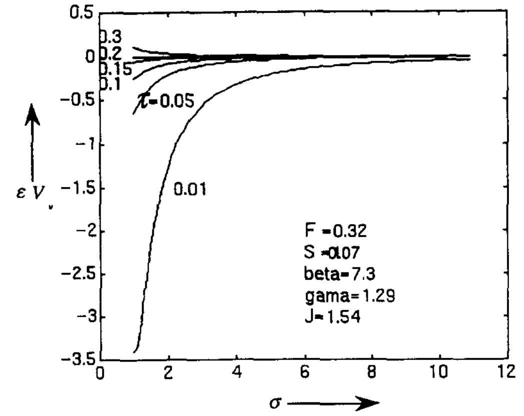

Fig. 3. Velocity in the vapor for different time τ.

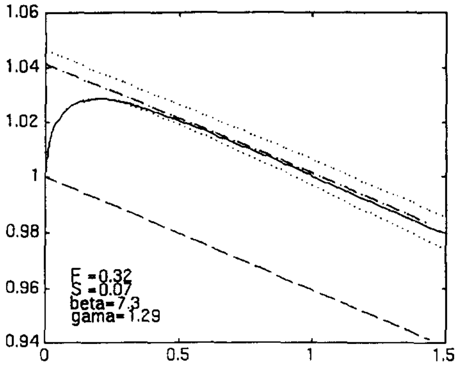

Fig. 4. Square of the radius of the drop vs time [plain line is the numerical computation, … is the inner expansion (98), -- is the outer expansion order one, ... is the outer expansion order two (101)–(103) and –· is the outer expansion order three].

radius computed analytically; we used the inner expansion (91), and outer expansion (101). The matching has been done at order one, that is $f(S) = 1$, at order two, $f(S) = 1+f_1 S$, and three $f(S) = 1+f_1\ S+f_2 S^2$. We notice that the outer expansion is a very good approximation from the reduced time at 0.5, the inner approximation describes correctly the variation from initial time up to the reduced time 1.

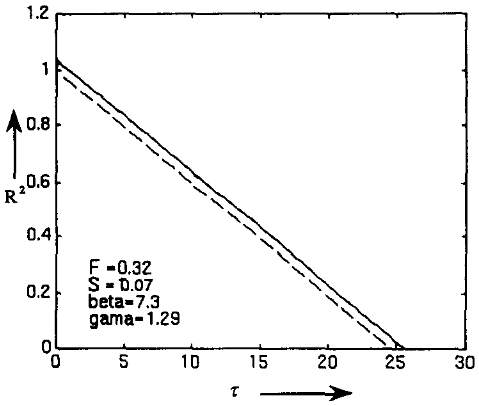

Fig. 5. Square of the radius vs time (plain line is the numerical computation and outer expansion at order two and the dashed line -- is the outer expansion at order one).

Let us now see the quality of the expansions for a long time period. In Fig. 5 we have plotted the evolution of the square of the radius computed numerically and analytically. We used outer expansion at order two only, i.e. with $f(S) = 1+f_1(S)$, at this scale of time the curve obtained by numerical computation and the one obtained analytically are superposed.

The difference in the intersection of the two curves with the time axis provides the error in the evaporation time given in equation (105).

We have, for the Mach number expansion, neglected the terms containing G and H as factors, because both of these numbers are proportional to M^2. However in equation (32), the concerned terms are proportional to the velocity of the interface which tends to infinity. We must check that all the terms remain finite. The first condition is

$$G(1-\rho_1^2)\frac{(R\dot{R})^3}{R^2} \leqslant \frac{1}{S}|R\dot{R}|,$$

that is $\sqrt{SG(1-\rho_1^2)}(\dot{\mathscr{P}}/2) \leqslant R$, which gives numerically $(2.257/2)\dot{\mathscr{P}} \leqslant R$. If we choose for $\dot{\mathscr{P}}$ the value obtained in equation (100), that is $\dot{\mathscr{P}} = 4.06\times 10^{-2}$, we obtain the $R \geqslant 4.5\times 10^{-2}$. The droplet is almost completely evaporated, and the radius is too small to apply the hypothesis of a continuum medium. The second term must be evaluated, and the condition can be written as

$$\rho_1 H\left|\frac{\partial V_{\rm v}}{\partial\sigma} - \frac{1-\rho_1}{2}\frac{\dot{\mathscr{P}}}{R}\right||\dot{R}| \leqslant \frac{|\dot{R}|R}{S}.$$

Using relation (59) we can evaluate the derivative of the velocity at the interface, it is $(\partial V_{\rm v}^0/\partial\sigma)|_{\sigma=1} \approx 5400(\dot{\mathscr{P}}^0/R)$. This relation with the previous inequality gives the condition $R \geqslant 2.4\times 10^{-5}$. Once again the term considered can be neglected for the same reasons.

5. CONCLUSION

The problem of the evaporation of a droplet in its vapor had been previously considered on the basis

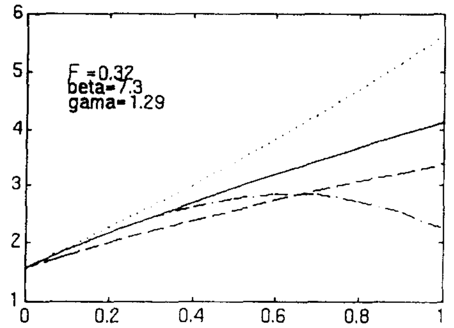

Fig. 6. Evaporation τ_{ev} time vs S (104) [plain line is the numerical computation, – – is the expansion order one for $f(S)$, . . . is the expansion order two for $f(S)$ and – · – · is the outer expansion order three for $f(S)$].

of the quasi-permanent hypothesis which consists of assuming that the temperature of the drop depends on time only by the variation of the radius. With this hypothesis it is possible to derive that the square of the radius is a linear function of time; this law known as the R^2 law is well shown by experimental results. We have reconsidered this old problem in the light of asymptotic expansion to determine the range of validity of the quasi-permanent hypothesis, and to improve, if possible, the R^2 law. We have seen that the quasi permanent hypothesis lies on the existence of three small parameters: the Mach umber, the ratio of the thermal diffusivities, and the Stefan number. All the expansions associated with these numbers are singular, but it is possible to obtain an expression which contains the previous one and correct the known law. The comparison with numerical computations shows that a matched expansion at second order, i.e. linear in the Stefan number, gives a quite good approximation for the evolution law of the drop radius. To appreciate the expansions dependence on S, we have plotted, on Fig. 6, the evaporation time τ_{ev} vs S, for the computed solution and the solutions obtained by expansion at different orders.

The expansion at order 3 is valid for $S < 0.4$; this interval gives the validity domain of the quasi-permanent hypothesis.

Let us note that the method applied here could be used for the combustion theory of droplets. The results would be similar for the drop radius, but not for the radius of the flame, the hypothesis of the analysis given here is not satisfied, the evolution of the flame radius never reaching a quasi-permanent regime, see Law [11].

Let us add, that none of the expansion in ε and S is uniform with respect to the other, consequently the order of expansion (firstly in ε secondly in S) is essential.

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